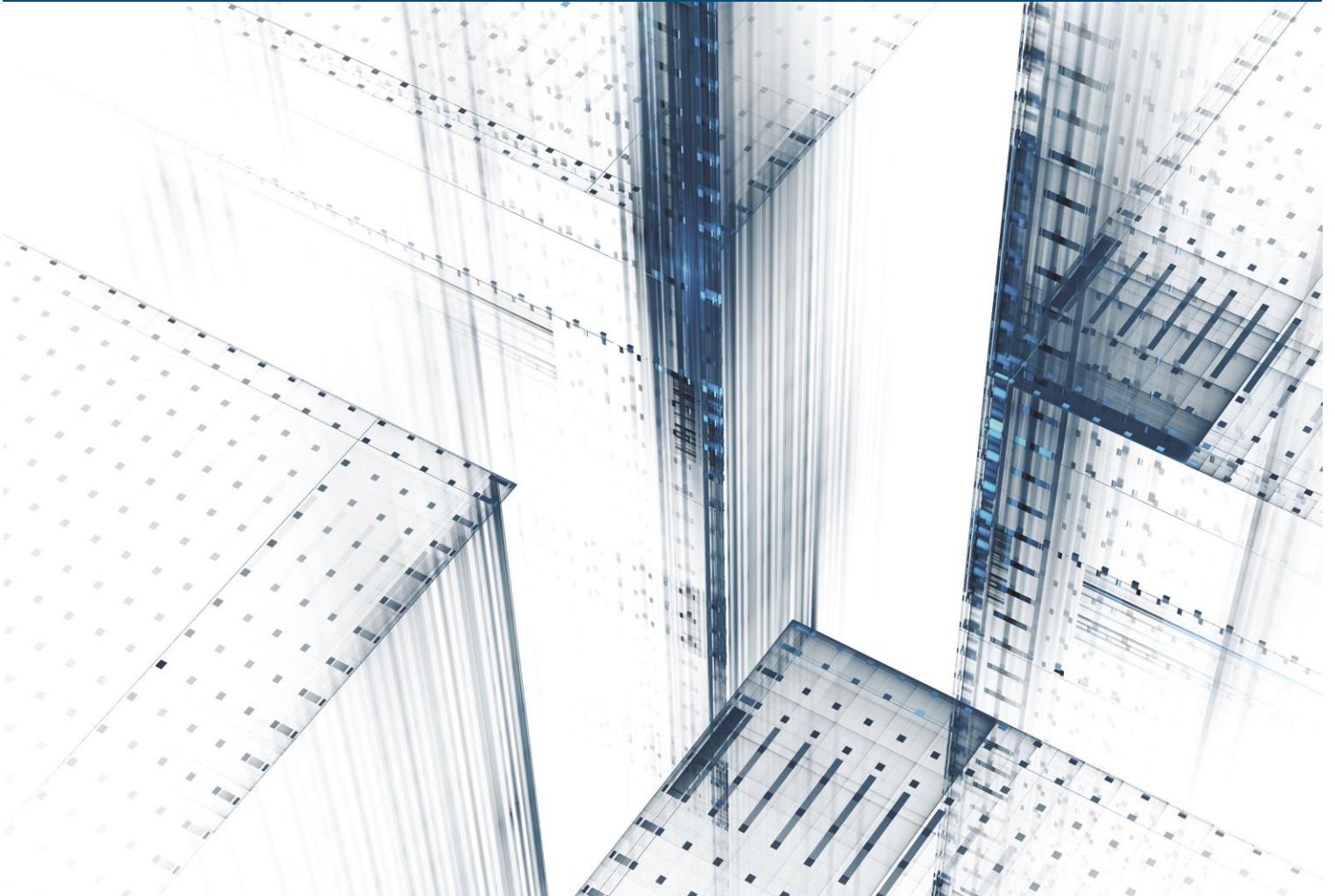




Parameter Uncertainty



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Parameter Uncertainty

In all fields of insurance and risk management, understanding the uncertainty associated with model estimates is essential to properly quantifying risk. In this report we examine and describe parameter uncertainty, providing simple structures to incorporate parameter uncertainty in a wide range of problems. We begin in section 1 with a review of the current literature. We look at work from many areas of actuarial science and draw connections to statistics and other applied fields with similar concerns. We continue in section 2 with a mortality and life insurance case study wherein we show a simple way to add parameter uncertainty to mortality rates. We compare the impact of parameter uncertainty on the present value of a simple life annuity. Next, in section 3, we estimate diabetes progression using an OLS regression. We show how to incorporate spike and slab priors to automatically perform model selection while fitting the model. We discuss how to choose good hyperparameters for the prior distributions and compare the impact of different choices on the model selection process. Finally, in section 4 we use Poisson regression to analyze claim counts from a set of auto insurance data from California. In this context, we compare different prior assumptions and show the value of including parameter uncertainty.

Section 1: Literature Review

The exploration of parameter uncertainty began in earnest with two papers, Heckman and Meyers (1983) and Meyers and Schenker (1983). Both examine the issue when modeling aggregate losses. Parameterized loss distributions (collective risk models) such as the generalized Poisson can make too many simplifying assumptions. They propose adding extra variability to the frequency and severity distributions to make the model more realistic. They focus on the following claim count distribution:

$$N \sim \text{Pois}(\lambda\chi),$$

$$\text{where } E(\chi) = 1 \text{ and } \text{Var}(\chi) = c.$$

They refer to c as the contagion parameter, which controls the degree of uncertainty. As $c \rightarrow 0$, the model becomes the standard Poisson claim count model. The severity distribution is then scaled by the parameter β where

$$E\left(\frac{1}{\beta}\right) = 1 \text{ and } \text{Var}\left(\frac{1}{\beta}\right) = b.$$

They show scenarios where the variance of the aggregate loss distribution is a function of the variance of both the count and severity distributions and b and c .

Meyers and Schenker (1983) add parameter uncertainty to expand a previous model using characteristic functions to understand the aggregate losses. Heckman and Meyers (1983) focus on the loss ratio, $X/E(X)$. They show that the variance of this ratio depends on the values of b and c , and they provide formulas for estimating b and c based on loss ratios from observed data. Furthermore, in connection with much of the credibility literature, they calculate b and c when the aggregate losses come from different groups who each have potentially different count and severity distributions.

Meyers (2007) uses many of the principles of parameter uncertainty from these papers and applies them to stochastic loss reserving. A Bayesian approach to modeling the claim counts and frequencies is compared with maximum likelihood estimates on a large number of insurers and shows that predictions can be improved using parameter uncertainty models.

In a slight extension of Heckman and Meyers (1983), Hayne (1999) comes to the conclusion that "parameter uncertainty and model or specification uncertainty are more significant issues to insurers than simple process uncertainty."

Wacek (2005) builds upon the loss ratio work of Heckman and Meyers (1983). He focuses on a standard Bayesian analysis of normal and lognormal data. Wacek uses posterior distributions of the parameters given the data to show that for a given sample size, the probability that the true parameters are within a reasonable range of the estimate are small, with larger sample sizes providing more accurate results. He focuses on a wide range of quantities of interest and compares the results with and without

parameter uncertainty. In addition to loss ratios, he examines aggregate excess reinsurance, downside risk measures and commissions.

Borowicz and Norman (2006) focus on aggregate losses above a certain level (the extreme values). They assume the count (N) follows a Poisson distribution and severity (X) follows a generalized Pareto distribution. Three different methods are compared to quantify the parameter uncertainty: (1) Fisher's information, (2) bootstrapping and (3) Bayesian methods. Diers, Eling and Linde (2013) build upon that work, using the same three methods to quantify parameter uncertainty. They also explore the effect of sample size on the results. They focus on aggregate claims and separate the large claims from the attritional claims. The large claims are given a Poisson count and Pareto severity while the attritional claims are lumped in one gamma distribution. They compare how some significant quantities of interest are affected by the parameter uncertainty in both a one-year time frame and a five-year time frame, the most significant of which is $\text{VaR}_{99.5}$.

In a similar vein, Gerrard and Tsanakas (2011) model failure where failure is a specific risk factor exceeding a certain threshold. The losses are random, and the risk capital is also random and initially held to be a percentile of the loss distribution. The probability of insolvency is calculated for a variety of loss distributions under maximum likelihood estimates. To decrease the probability of insolvency in the presence of uncertainty, they explore setting the risk capital to a higher percentile of the loss distribution than actually desired. The main argument of the paper is that instead of setting the percentile higher, one can introduce parameter uncertainty via a Bayesian prior into the loss distribution and then use the desired percentile. The prior is integrated out when the percentile is calculated.

Bignozzi and Tsanakas (2015) focus more generally on how the risk measures are affected by parameter uncertainty. As expected, the required capital to stay within certain risk requirements is larger when parameter uncertainty is present. They focus on location-scale families and show that the adjustment to the risk measure is mathematically convenient. They use a measure called the residual risk, which, for a given risk metric such as VaR, capital estimator and loss random variable, returns a value that evaluates the risk in the presence of additional uncertainty. A good goal is for this to be 0. They show that when a loss distribution is used with a Bayesian prior integrated out, the capital required for the residual risk to be zero is larger than when parameter uncertainty is not required.

Taking a broader view, Cairns (2000) discusses model and parameter uncertainty. The approach is inherently Bayesian, and the parameter uncertainty is incorporated through a prior on the parameters. He reviews three statistical methods of model uncertainty: (1) comparing model fit using a criterion such as AIC or BIC, (2) finding the model from a set of models with the highest posterior probability and (3) finding the posterior distribution of the parameters of all the models along with the posterior probabilities of the specific models. He includes two examples. In the first, he simulates stochastic interest rates using random rates of interest with continuous force between time points. He then uses two models, an i.i.d. normal model and an AR(1) model, and compares them using all three methods of model uncertainty. In his second example, he simulates data from a lognormal distribution and compares 10 gamma likelihoods with different scale parameters. From the resulting fit he then calculates the amount needed to hold in reserve to avoid ruin and determines that adding parameter uncertainty requires more to be held in reserve.

Additionally, many papers look at parameter uncertainty in specific applications. Cairns, Blake and Dowd (2006) is the first in a long list of papers exploring parameter uncertainty in stochastic mortality. Mortality is modeled as a logistic regression with parameters that vary stochastically across time. Parameter uncertainty comes from priors on the mean and the variance terms of the logistic regression parameters, with a multivariate normal on the mean vector and inverse Wishart on the covariance matrix. The difference between the point estimates of the two models (with or without parameter uncertainty) is minimal.

In financial risk management, Hartman and Heaton (2011) examine the impact of parameter uncertainty in asset simulation, especially regime-switching models. They use a standard Bayesian model to add parameter uncertainty and a Bayesian nonparametric model to account for model uncertainty. They found that incorporating parameter uncertainty had a much bigger impact on the risk measures than the model uncertainty. These models were further extended beyond regime-switching models to GARCH and stochastic volatility models in Hartman and Groendyke (2013).

In health insurance, Fellingham, Kottas and Hartman (2015) examine the impact of parameter and model uncertainty on group health claim prediction. They find that a model that incorporates both types of uncertainty outperforms other models when

predicting future year claims. The improvement is greatest when predicting claims for a new group that is not in the current data set.

In property and casualty insurance, Shi and Hartman (2016) examine the impact of parameter uncertainty in loss reserving. They build on the classic chain-ladder model using Bayesian regression. They also show how to incorporate information from similar lines of business to improve predictive accuracy. Venter and Sahasrabuddhe (2015) review various ways to incorporate parameter uncertainty (which they call parameter risk).

A number of actuarial applications have been analyzed using parameter uncertainty in the framework of Bayesian methodology. Shi, Basu and Meyers (2012) model multivariate reserving from multiple lines of insurance. They use a Bayesian log-normal model and directly model the pairwise correlations between the different lines. They show that by employing valuation actuaries' prior knowledge they can significantly improve prediction over the model without parameter uncertainty. Meyers (1999) also estimates correlation between multivariate outcome variables using parameter uncertainty in predictive modeling.

The idea of parameter uncertainty being used to make more realistic models is employed in various other fields as well, namely, science and engineering (e.g., Kitanidis (1986), Wood and Rodriguez-Iturbe (1975), Beck and Katafygiotis (1998)). An entire field in statistics has arisen to address the nature and degree of parameter uncertainty seen in certain physical models. This field has been termed uncertainty quantification (Smith, 2013). Much of the parameter uncertainty that will arise in actuarial applications can be addressed using Bayesian methodology. References such as Gelman et al. (2014) and Marin and Robert (2007) share insight on how to set up Bayesian models, how to analyze them and how to answer questions using them.

Section 2: Life Case Study

In this case study we will examine the impact of parameter uncertainty on mortality estimates and the price of a single premium immediate annuity (SPIA). Throughout the case study, we will use the 2012 IAM mortality tables as our industry table. We will focus on the table for females in this case study, which has the mortality rates shown in Table 1.

We will examine two different ways to incorporate parameter uncertainty into our mortality models. We will first show how to model the ratio of our actual to tabular rates. Afterward, we will model the mortality rates directly. Modeling the ratio of actual to tabular rates allows us to use all the data together but is very constraining on the shape of the mortality curve. Modeling each of the mortality rates individually is at the other extreme. The age x mortality rate uses only age x data, but the shape of the curve is entirely flexible. For both methods we will focus on rather simple cases, and both can be extended to remove some of their shortcomings. Those extensions are briefly discussed in the conclusion.

2.1 ACTUAL-TO-TABULAR RATIO

Company-level data will often be expressed as a factor multiplied by the industry table, an actual-to-tabular ratio. We can model and incorporate parameter uncertainty directly into the ratio. The likelihood of our data is

$$Y_a \sim \text{Bin}(E_a, \theta T_a),$$

where Y_a is the number of policyholder deaths of age a , E_a is the number of policyholders of age a , T_a is the tabular mortality rate and θ is the actual-to-tabular ratio. Notice that in our scenario, θ is constant for all ages. We set it up that way for clarity and simplicity, but it is not required. Later in this section, we will show how to relax that assumption.

To build in the parameter uncertainty, we will define a rather noninformative prior on θ :

$$\theta \sim N(1, 1^2).$$

Table 1
2012 FEMALE IAM BASIC TABLE

Age	1000qx2012	Age	1000qx2012	Age	1000qx2012	Age	1000qx2012
0	1.801	30	0.333	60	3.844	90	98.197
1	0.450	31	0.357	61	4.352	91	108.323
2	0.287	32	0.375	62	4.899	92	119.188
3	0.199	33	0.390	63	5.482	93	131.334
4	0.152	34	0.405	64	6.118	94	145.521
5	0.139	35	0.424	65	6.829	95	162.722
6	0.130	36	0.447	66	7.279	96	182.120
7	0.122	37	0.476	67	7.821	97	199.661
8	0.105	38	0.514	68	8.475	98	217.946
9	0.098	39	0.560	69	9.234	99	236.834
10	0.094	40	0.613	70	10.083	100	256.357
11	0.096	41	0.667	71	11.011	101	283.802
12	0.105	42	0.723	72	12.030	102	304.716
13	0.120	43	0.774	73	13.154	103	325.819
14	0.146	44	0.823	74	14.415	104	346.936
15	0.174	45	0.866	75	15.869	105	367.898
16	0.199	46	0.917	76	17.555	106	387.607
17	0.220	47	0.983	77	19.500	107	400.000
18	0.234	48	1.072	78	21.758	108	400.000
19	0.245	49	1.168	79	24.412	109	400.000
20	0.253	50	1.290	80	27.579	110	400.000
21	0.260	51	1.453	81	31.501	111	400.000
22	0.266	52	1.622	82	36.122	112	400.000
23	0.272	53	1.792	83	41.477	113	400.000
24	0.275	54	1.972	84	47.589	114	400.000
25	0.277	55	2.166	85	54.441	115	400.000
26	0.284	56	2.393	86	61.972	116	400.000
27	0.290	57	2.666	87	70.155	117	400.000
28	0.300	58	3.000	88	78.963	118	400.000
29	0.313	59	3.393	89	88.336	119	400.000
						120	400.000

The results are sensitive to the size of the variance in the prior. If the variance is set much smaller than 1, the prior mean will have an impact on the results. The impact will increase as the variance is set to smaller levels. This is desired if the prior is used to incorporate expert opinion, but for our example we are assuming to outside information and just allowing the data to speak.

Unfortunately, the conditional posterior distribution of θ does not have a closed form. We do not know the normalizing constant, but we do know what it is proportional to:

$$f(\theta|Y, E, T) \propto e^{-\frac{(\theta-1)^2}{2 \cdot 0.02^2}} \cdot \prod_a \binom{E_a}{Y_a} (\theta T_a)^{Y_a} (1 - \theta T_a)^{1-Y_a}.$$

We will use Markov chain Monte Carlo (MCMC, specifically a Metropolis-Hastings sampler) to simulate draws from the posterior distribution of θ . We will propose new values of θ from a normal distribution centered at the current value of θ with a small standard deviation (we will explore the sensitivity of our results to the choice of standard deviation later). The steps of the algorithm are as follows:

1. Draw an initial value for θ , $\theta^{(0)}$, from the prior distribution or set it at an arbitrary value.

2. Until you reach the desired number of iterations, for each iteration i
 - a. Propose a new value of θ , θ^* , drawn from the proposal distribution, $N(\theta^{(i-1)}, 0.035^2)$.
 - b. Calculate the Metropolis-Hastings acceptance probability, α , using the following equation:

$$\alpha = \frac{f(\theta^*|Y, E, T)}{f(\theta^{(i-1)}|Y, E, T)}.$$

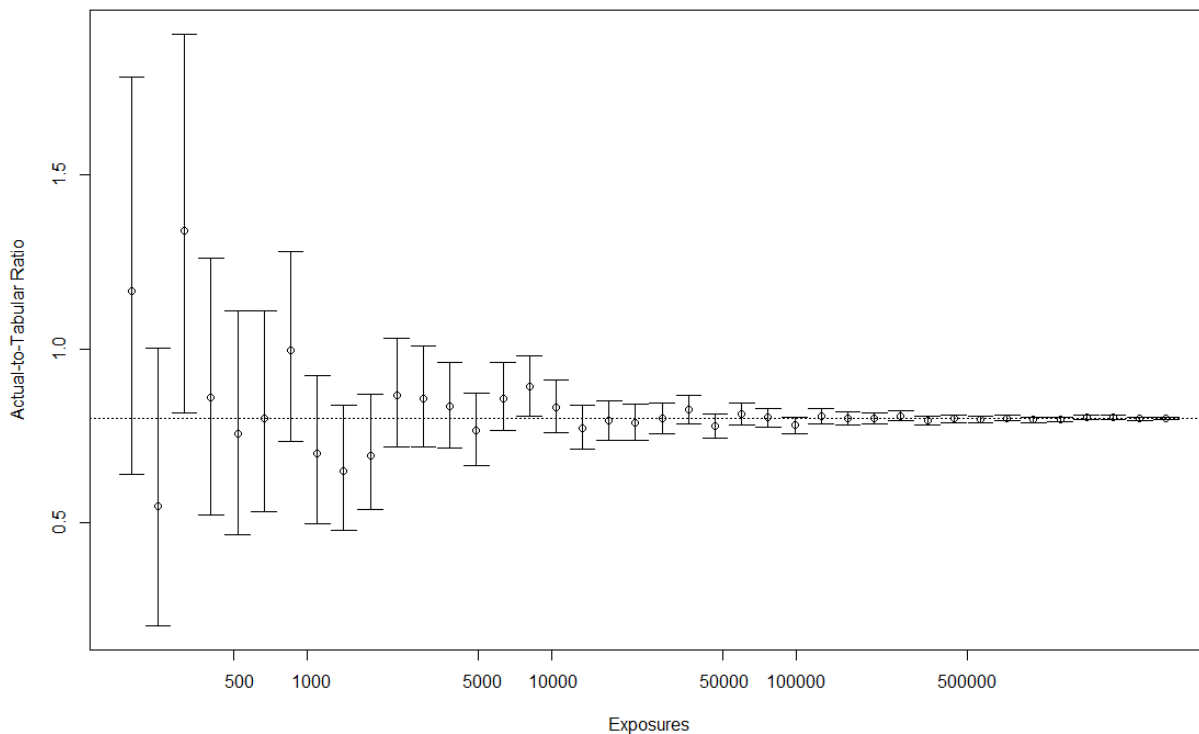
Note that even though we don't know the normalizing constant of the posterior distribution, they will cancel in the acceptance probability.

- c. Draw $U \sim Unif(0,1)$. If $\alpha > U$, then accept the proposed value and $\theta^{(i)} = \theta^*$. Otherwise, you reject the proposed value and $\theta^{(i)} = \theta^{(i-1)}$.
3. Use the draws (after removing the first few for burn-in) to understand the posterior distribution of θ .

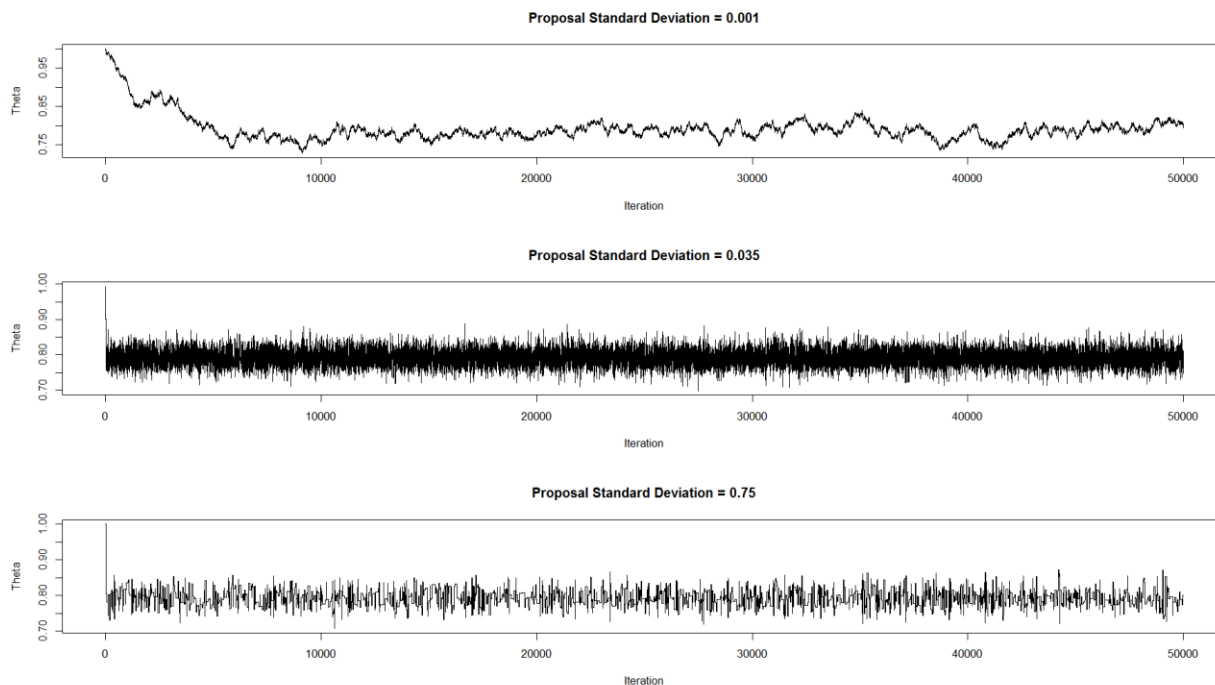
2.1.1 POSTERIOR DISTRIBUTION FOR VARYING SAMPLE SIZES

The posterior distribution of the actual-to-tabular rate will depend heavily on the sample size of the company data. For illustration, we will assume that the company data follow mortality rates at 80% of the IAM basic female table at ages 65+. We simulate data with varying exposures to examine the impact of sample size on the posterior distribution of the actual-to-tabular ratio. The 95% credible intervals are plotted in Figure 1. When the sample size is small, there is still a lot of uncertainty about the ratio, up to 0.2 in total spread. As the sample grows, the uncertainty about the ratio shrinks and the central estimate converges on the true value.

Figure 1
ACTUAL-TO-TABULAR RATIO 95% CREDIBLE INTERVALS AS A FUNCTION OF EXPOSURES



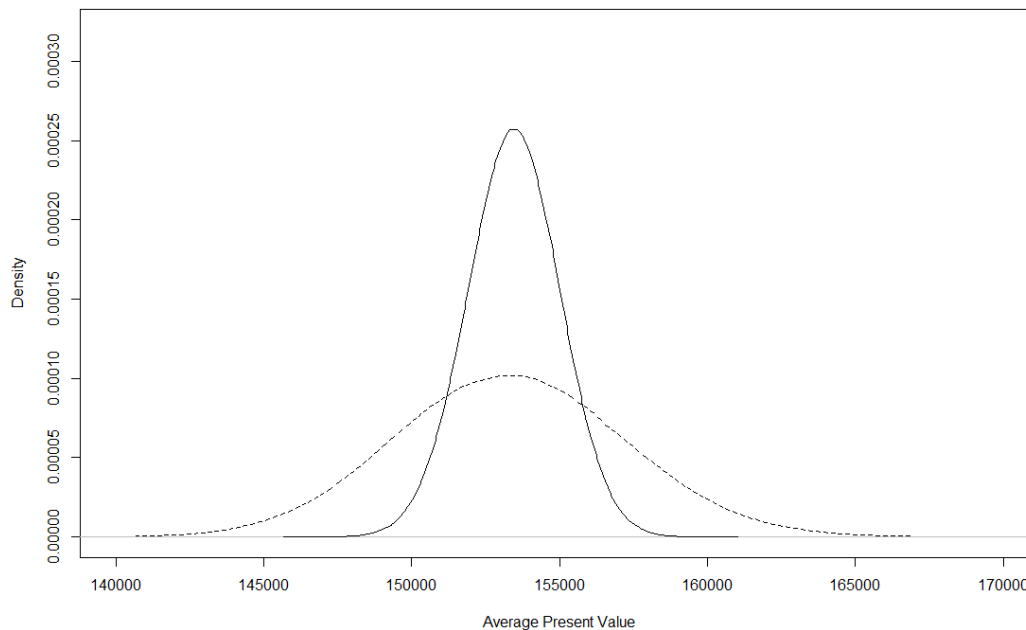
The proposal standard deviation of θ will impact how efficiently the MCMC algorithm will explore the posterior distribution. Pick a value that is too small and the chain will take a long time to get to all parts of the posterior. Pick one that is too large and your chain will reject proposals too often. Figure 2 shows several trace plots for different values of the proposal standard deviation. In our example, any value between about 0.1 and 0.45 seemed to work well.

Figure 2TRACE PLOTS OF θ AS A FUNCTION OF THE PROPOSAL STANDARD DEVIATION

2.1.2 SINGLE PREMIUM IMMEDIATE ANNUITY

Although it is important to understand the parameter uncertainty inherent in the ratio itself, we are more concerned with the effect of incorporating parameter uncertainty on the price, margins and reserves of a portfolio of contracts. We focus on a simple life insurance product, a single premium immediate annuity (SPIA). So that we can focus on the impact of adding parameter uncertainty to the mortality rates, for all the contracts we will assume a discount rate of 3%, an annual payment of 10,000 and a 65-year-old female insured. Using standard methods and the industry mortality rates, we estimate the actual-to-tabular ratio using maximum likelihood from a simulated portfolio of about 2,000 contracts. We find that the expected present value of this product is 153,444.90. To understand the uncertainty around this EPV, we will simulate the mortality experience of a portfolio of 1,000 contracts and calculate the present value of that portfolio and divide by 1,000 to get the average present value. We repeat that process for 100,000 portfolios. To incorporate parameter uncertainty, we obtain 100,000 draws from the posterior distribution of θ and then simulate each portfolio using a new value of θ . The density estimates of those present values are shown in Figure 3. When adding the parameter uncertainty, the variance of the expected value increases as expected.

Figure 3
PRESENT VALUE OF SPIA CONTRACTS BOTH WITH (DASHED LINE) AND WITHOUT (SOLID LINE) PARAMETER UNCERTAINTY.



2.1.3 EXTENSIONS

Although it may be sufficient to have a single actual-to-tabular ratio, it is relatively straightforward to extend the model to allow the ratio to differ by age, gender, product, underwriting or any other subgroup. Our current model is

$$Y_a \sim \text{Bin}(E_a, \theta T_a),$$

$$\theta \sim N(1, 1^2).$$

To allow θ to vary by a subgroup, such as gender and age, the model can be adjusted as follows:

$$Y_{ag} \sim \text{Bin}(E_{ag}, \theta_{ag} T_{ag}),$$

$$\theta_{ag} \sim N(\mu + \tau_g + \nu_a, 0.02^2),$$

$$\mu \sim N(1, 1^2),$$

$$\tau_g \sim N(0, 0.1^2),$$

$$\nu_a \sim N(0, 0.1^2).$$

Under this model, the gender effect is captured through τ_g , the age effect through ν_a , and μ is an overall mean or intercept. To estimate the parameters in this model, you could sequentially update the parameters using the following conditional distributions:

$$f(\theta|\mu, Y, E, T) \propto e^{-\frac{(\theta-1)^2}{2 \cdot 0.02^2}} \cdot \prod_{a,g} \binom{E_{ag}}{Y_{ag}} (\theta T_{ag})^{Y_{ag}} (1 - \theta T_{ag})^{1-Y_{ag}},$$

$$\mu|\theta, Y, E, T, \tau, \nu \sim N\left(\frac{1 + \frac{\sum_{a,g} \theta - \tau_g - \nu_a}{0.02^2}}{1 + \frac{1}{0.02^2}}, \frac{1}{1 + \frac{1}{0.02^2}}\right),$$

$$\tau|\theta, Y, E, T, \mu, \nu \sim N\left(\frac{\frac{1}{0.1^2} + \frac{\sum_{a,g} \theta - \mu - \nu_a}{0.02^2}}{\frac{1}{0.1^2} + \frac{1}{0.02^2}}, \frac{1}{\frac{1}{0.1^2} + \frac{1}{0.02^2}}\right),$$

$$\nu|\theta, Y, E, T, \mu, \tau \sim N\left(\frac{\frac{1}{0.1^2} + \frac{\sum_{a,g} \theta - \mu - \tau_g}{0.02^2}}{\frac{1}{0.1^2} + \frac{1}{0.02^2}}, \frac{1}{\frac{1}{0.1^2} + \frac{1}{0.02^2}}\right).$$

We would only need to use a Metropolis-Hastings step on θ ; the rest can be directly drawn from known distributions. The model can be further extended to include other categories, interactions between the groups (allowing the gender effect to differ by age) and correlations between the ages (to constrain the ν_a to change smoothly across ages), among many other options.

2.2 MODELING THE MORTALITY RATES DIRECTLY

To incorporate parameter uncertainty directly and be able to update the industry estimates with our internal data, we will assume that the mortality rates follow a Beta distribution a priori:

$$q_x \sim \text{Beta}(\alpha, \beta),$$

$$f(q_x) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} q_x^{\alpha-1} (1 - q_x)^{\beta-1}, \quad \alpha > 0, \quad \beta > 0.$$

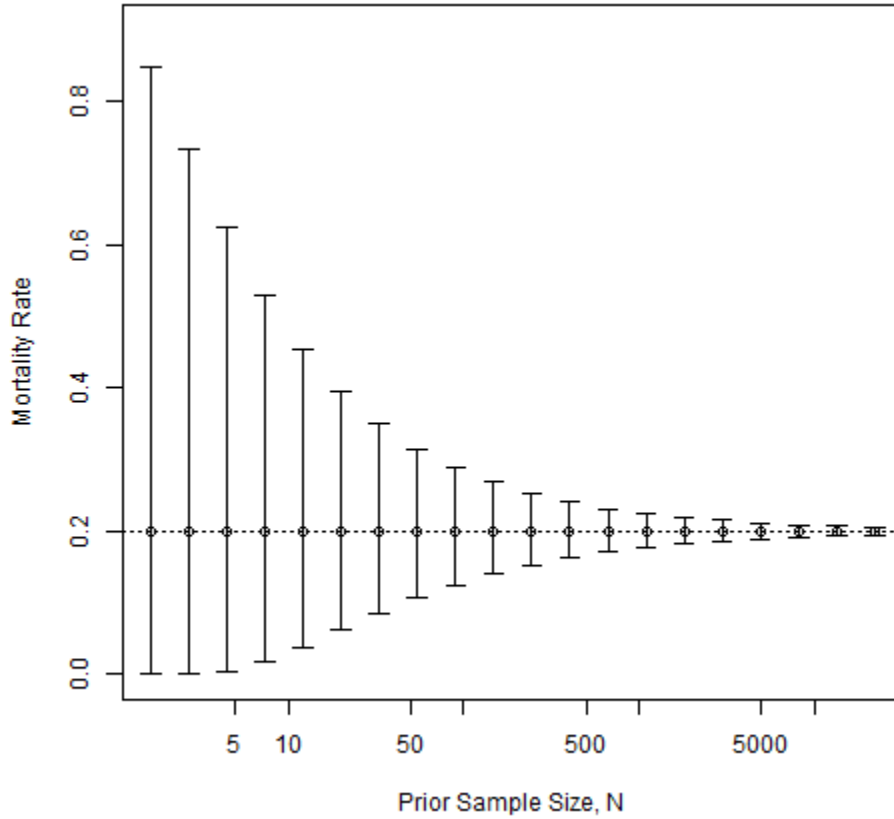
We chose the Beta distribution because it is constrained to be between zero and one and it is the conjugate prior when dealing with binomial responses. Assuming that your data have X successes (in this case, deaths) out of a possible E trials (lives of age x), then the posterior distribution will also follow a Beta distribution with the following parameters:

$$q_x|E, X \sim \text{Beta}(\alpha + X, \beta + E - X).$$

2.2.1 SINGLE MORTALITY

The prior distribution is centered on the industry table value. The spread of the distribution is determined by the parameter N , which is the “prior sample size.” The prior is given weight equal to a data set of size N with Nq_x^{ind} deaths where q_x^{ind} is the industry table mortality rate. If the industry data come from a population that perfectly matches your company data (for all the characteristics that affect mortality), N could be equal to the number of exposures at each year in the industry study. More likely, the industry data will differ from your book in meaningful ways, so you will need to reduce N . At the other extreme, if the populations differ so greatly that you do not want to use much of the industry data at all, you can make N very small or even 0. The prior sample size can also vary by age, gender or any other characteristic you prefer. As N gets larger, the prior distribution becomes narrower, meaning that you are more confident that the mortality rate is close to the industry rate. The more confident you are about the prior (industry) mortality rate, the smaller the impact of the data. For a simple example, we will examine a single mortality rate (Figure 4) where the industry mortality rate is 0.2. When N is 50, the credible interval (the set of reasonable values for the true mortality rate) is still rather large, about 0.2 in total width. When N grows to many thousands, the range of plausible values becomes very small.

Figure 4
 95% PRIOR CREDIBLE INTERVALS FOR THE MORTALITY RATE AS A FUNCTION OF PRIOR SAMPLE SIZE.



If the sample mortality rate of the company data matches the industry table, adding data to the prior will increase confidence in the mortality estimate (reduce the uncertainty around that estimate). For example, using the same prior distribution above, assume we have 1,000 lives age x of which 200 die. The posterior credible intervals are in

Figure 5.

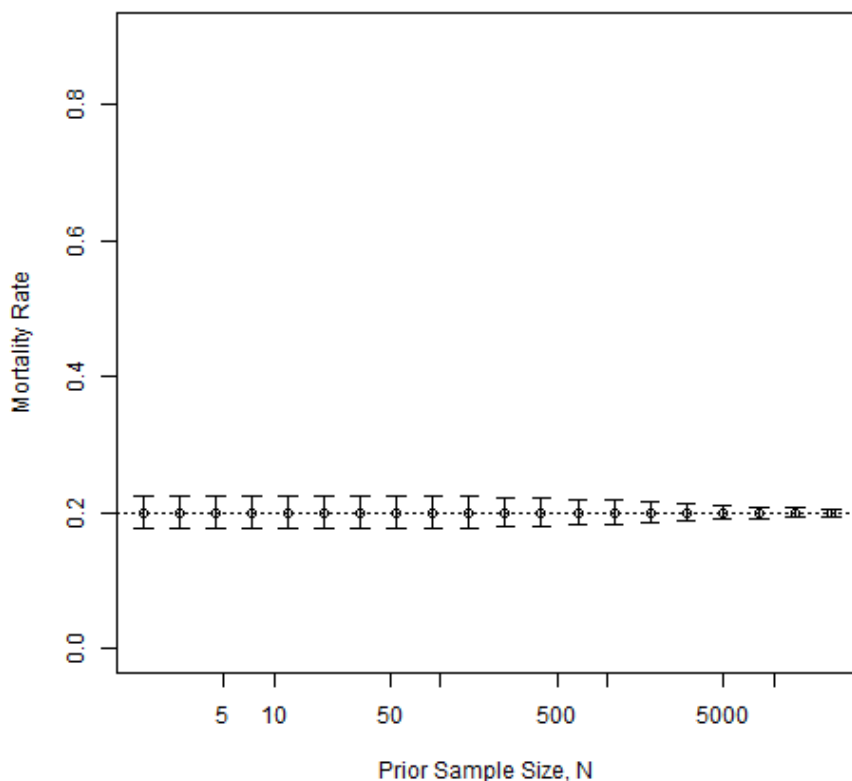
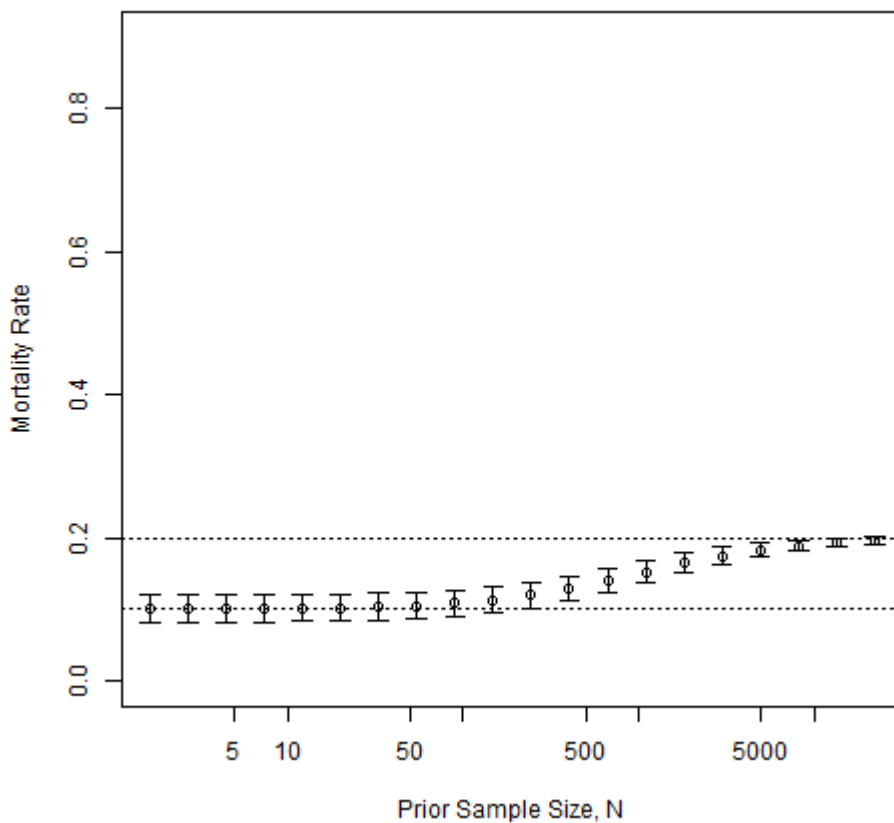


Figure 5
 95% POSTERIOR
 CREDIBLE
 INTERVALS FOR
 THE MORTALITY
 RATE AS A
 FUNCTION OF
 PRIOR SAMPLE
 SIZE WHEN THE
 OBSERVED DATA
 MATCH THE
 PRIOR
 ASSUMPTIONS.

Notice that the estimates stay the same ($\hat{q} = 0.2$) but the uncertainty around the estimates is reduced, most dramatically for the small prior sample size values. That makes intuitive sense because the difference between a sample of 10 and 1,010 is much greater than the difference between one of 5,000 and 6,000.

If the company data have a different mortality rate than the industry data, the impact on the posterior estimate depends heavily on the prior sample size. For the example below, we assume only 100 of your 1,000 observations died in the year in question. When the prior sample size is very small, the 1,000 data points completely overwhelm the prior, and the estimate is essentially 0.1. As the prior sample size grows, the posterior estimate becomes a weighted average of the prior estimate (0.2) and the sample mean (0.1). Also notice that as the prior sample size becomes much larger than the data set size, the posterior estimate is essentially the prior estimate (0.2). The 95% credible intervals are shown in Figure 6.

Figure 6
 95% POSTERIOR CREDIBLE INTERVALS FOR THE MORTALITY RATE AS A FUNCTION OF PRIOR SAMPLE SIZE WHEN THE OBSERVED DATA DIFFER FROM THE PRIOR ASSUMPTIONS

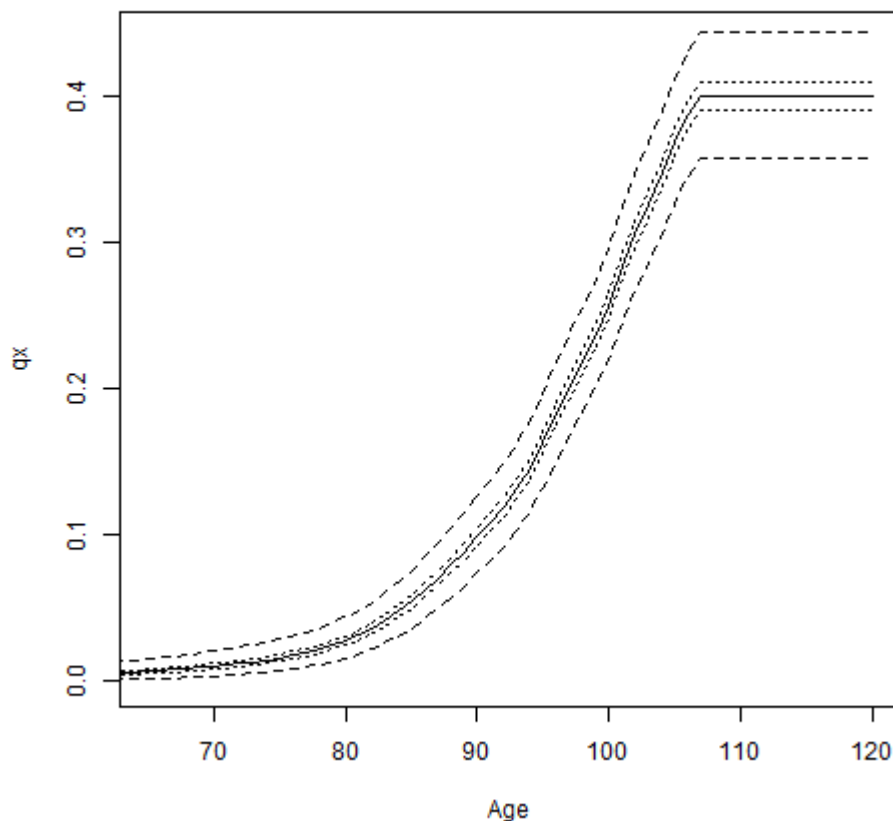


2.2.2 ENTIRE MORTALITY CURVE

The same effect can be seen when looking at the entire mortality curve. When looking at the prior densities,

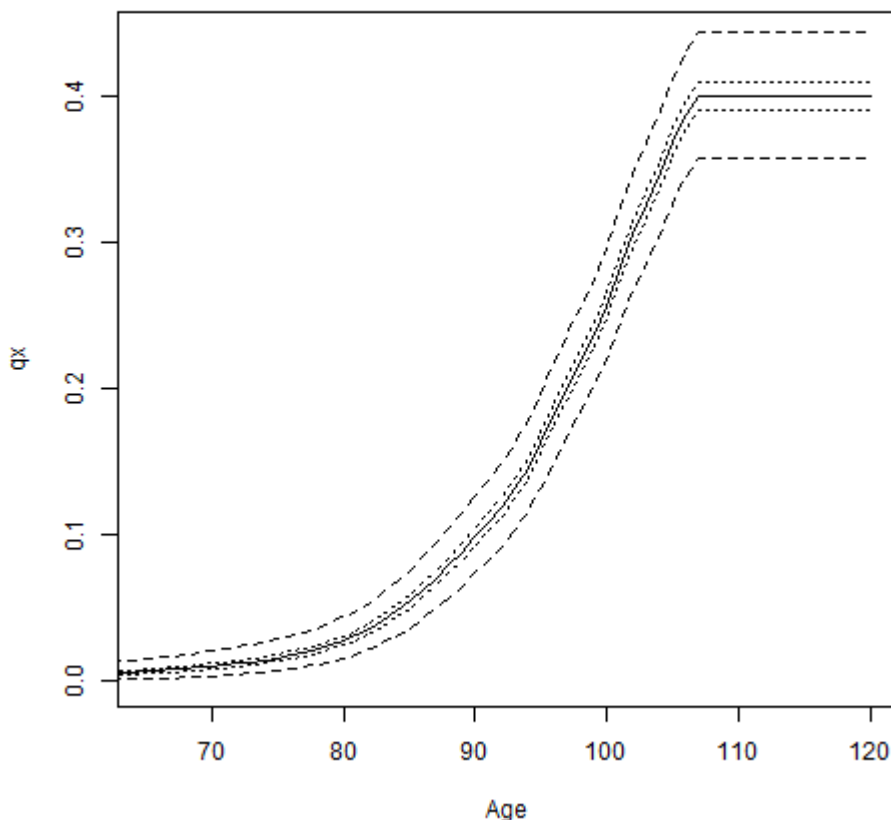
Figure 7

95% ELEMENT-WISE PRIOR CREDIBLE INTERVALS FOR THE ENTIRE MORTALITY CURVE. THE MORTALITY RATE IS THE SOLID LINE, THE 95% CREDIBLE INTERVAL WITH A PRIOR SAMPLE SIZE OF 500 IS THE DASHED LINE AND THE 95% CREDIBLE INTERVAL WITH A PRIOR SAMPLE SIZE OF 10,000 IS THE DOTTED LINE.



shows the industry mortality curve as the solid line, the 95% element-wise prior credible intervals with a prior sample size of 500 as the dashed line and a prior sample size of 10,000 as the dotted line. For simplicity, we assume that the prior sample size is set to a constant value regardless of age. This can be made completely flexible, allowing the prior sample size to decrease with the cohort as ages increase, or allowing the prior sample size to change as the similarity between the industry and company data increases and decreases. Similar to what we saw looking at a single mortality rate, the higher prior sample size has much less uncertainty around the table rates. Notice also that we used only the ages above 65 to make the plot cleaner and clearer and for easy application to our annuity example.

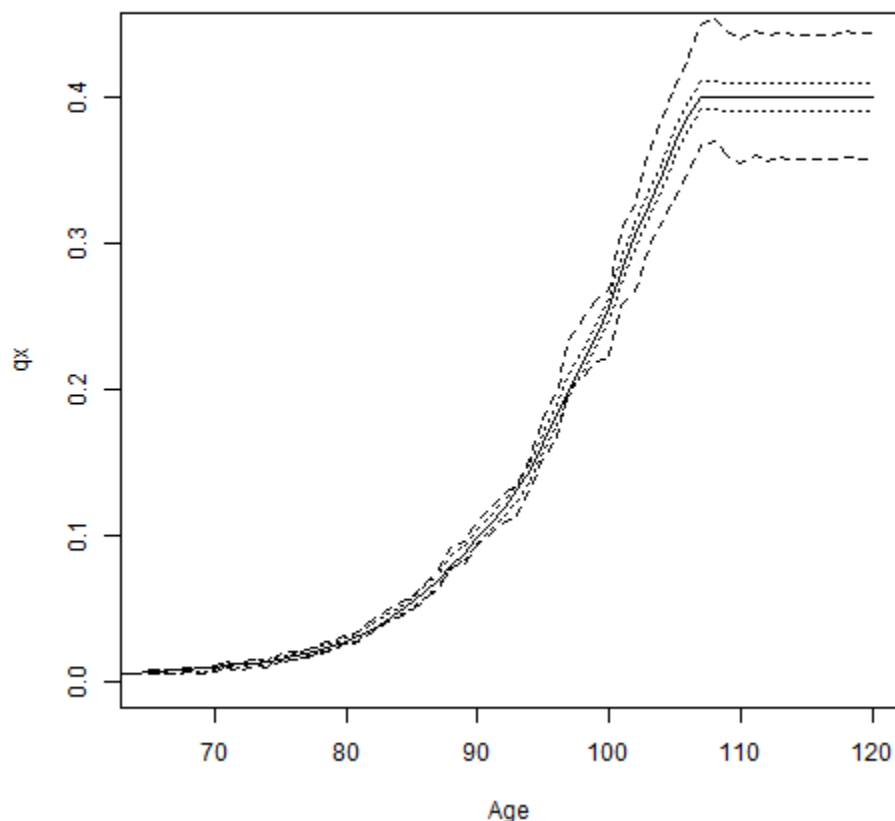
Figure 7
 95% ELEMENT-WISE PRIOR CREDIBLE INTERVALS FOR THE ENTIRE MORTALITY CURVE. THE MORTALITY RATE IS THE SOLID LINE, THE 95% CREDIBLE INTERVAL WITH A PRIOR SAMPLE SIZE OF 500 IS THE DASHED LINE AND THE 95% CREDIBLE INTERVAL WITH A PRIOR SAMPLE SIZE OF 10,000 IS THE DOTTED LINE.



Now, to show the impact of including company-specific data, we generate data at each individual age. The number of exposures at each age are proportional to the cohort size at that age assuming 10,000 started at age 65. The data are generated using the industry table, and the results are shown in Figure 8. The posterior credible intervals are more jagged and jittery because of the randomness in our data set. This also illustrates a shortcoming in this model: all the mortality rates are assumed to be independent (after accounting for the industry table correlations). The observed mortality rate at age 81 has no effect on the estimated mortality rate at age 80 or 82. Because all the lives started at age 65, only about 8% made it to age 100. Those 800 lives still have a large effect when the industry table is only given the impact of 500 lives (dashed lines), but they don't have much effect when the industry table is given the impact of 10,000 lives (dotted lines). The effect is rather large at the younger lives, making the width of the credible intervals much smaller.

Figure 8

95% ELEMENT-WISE POSTERIOR CREDIBLE INTERVALS FOR THE ENTIRE MORTALITY CURVE. DATA DRAWN FROM THE MORTALITY TABLE ARE INCORPORATED. THE MORTALITY RATE IS THE SOLID LINE, THE 95% CREDIBLE INTERVAL WITH A PRIOR SAMPLE SIZE OF 500 IS THE DASHED LINE AND THE 95% CREDIBLE INTERVAL WITH A PRIOR SAMPLE SIZE OF 10,000 IS THE DOTTED LINE.



2.2.3 SINGLE PREMIUM IMMEDIATE ANNUITY

We now incorporate parameter uncertainty in the mortality rates, following the methodology in Section 2.2.2. In Figure 9 we compare the effect of parameter uncertainty on the present values when the 10,000 observed lives follow the industry mortality. In Figure 10, we show what happens when the observed lives' mortality rates are 10% better. Note that in both these scenarios we are assuming no mortality improvement.

As expected, when the company data follow the industry table, adding parameter uncertainty simply increases the uncertainty in the present values. The expected present value remains the same. But, when the company data have better mortality experience than the industry table, it will pull the present values up. The pull is greatest when the prior sample size is smallest. Notice also that the smaller prior sample size has the larger posterior uncertainty.

Figure 9
 RANGE OF POSSIBLE PRESENT VALUES OF A 10,000 SPIA ISSUED TO AGE 65 WITH A DISCOUNT RATE OF 3%. THE SOLID LINE HAS NO MORTALITY RATE UNCERTAINTY, THE PRIOR SAMPLE SIZE OF 500 IS THE DASHED LINE AND THE PRIOR SAMPLE SIZE OF 10,000 IS THE DOTTED LINE. THE DATA ARE DRAWN FROM THE INDUSTRY TABLE.

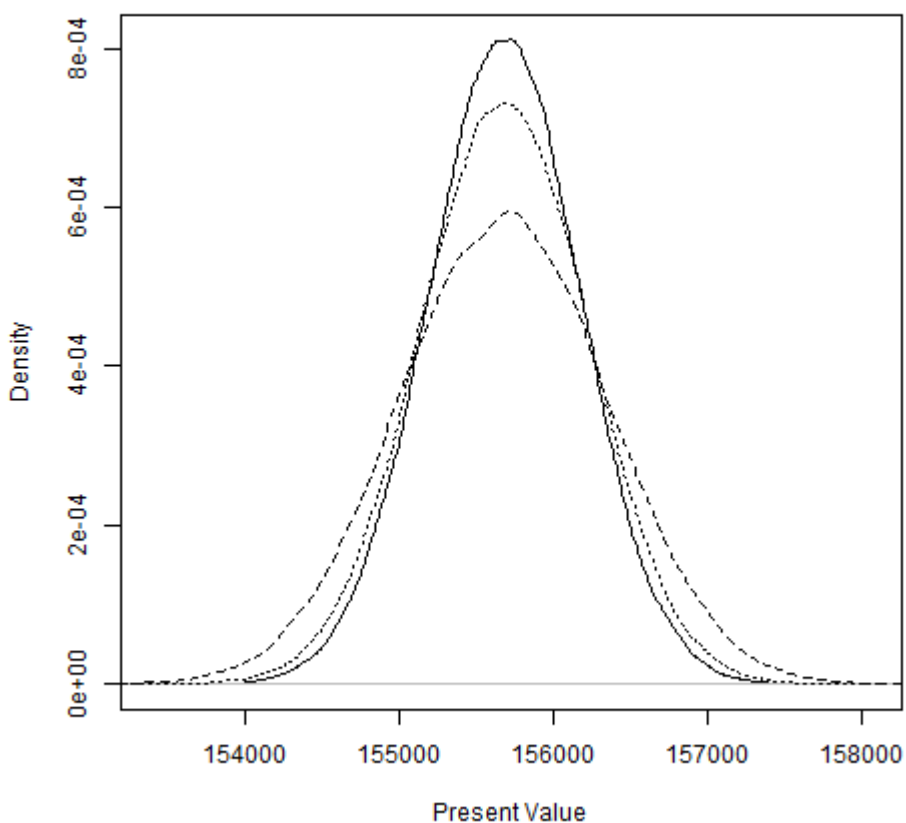
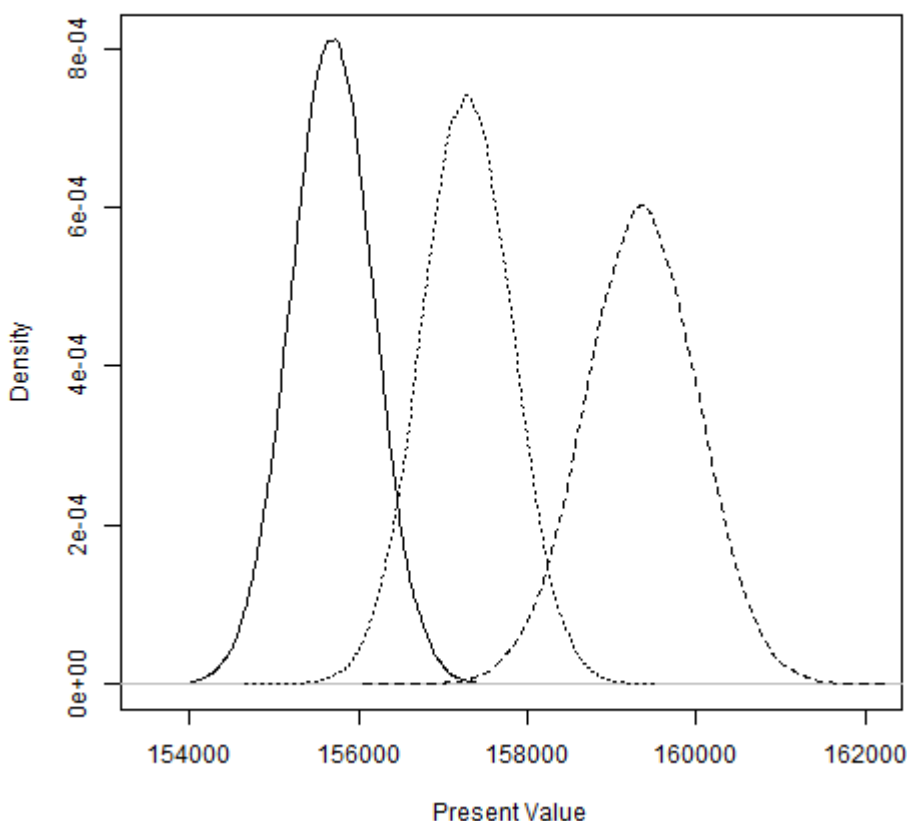


Figure 10
 RANGE OF POSSIBLE PRESENT VALUES OF A 10,000 SPIA ISSUED TO AGE 65 WITH A DISCOUNT RATE OF 3%. THE SOLID LINE HAS NO MORTALITY RATE UNCERTAINTY, THE PRIOR SAMPLE SIZE OF 500 IS THE DASHED LINE AND THE PRIOR SAMPLE SIZE OF 10,000 IS THE DOTTED LINE. THE DATA ARE DRAWN FROM MORTALITY BETTER THAN THE INDUSTRY TABLE.



2.3 TAIL MEASURES

To further explore the impact of parameter uncertainty on the annuity product, we compare the tail quantiles under the various models. Specifically, we calculate the 75%, 90%, 95%, 99%, 99.5% and 99.9% quantiles for each of our four models, no parameter uncertainty, modeling the actual-to-tabular ratio (Section 2.1) and modeling the individual mortality rates (Section 2.2) with $N = 500$ and $N = 10,000$. The results are presented in Table 2.

When comparing the individual mortality rate model to the model without parameter uncertainty, the difference between the quantile estimates increases only gradually as the quantile increases. Because the mortality rates are assumed to be uncorrelated, the tail events will not be as extreme. The opposite occurs when modeling the actual-to-tabular ratio. Because all the rates are perfectly correlated (they are all multiplied by the same random variable), tail events will be more extreme. Also, the actual-to-tabular tail quantiles are larger than those from both the individual models. Finally, because of the increased uncertainty when N is smaller, the model with $N = 500$ has uniformly larger tail quantiles than the model with $N = 10,000$.

Table 2
TAIL QUANTILES

	75%	90%	95%	99%	99.5%	99.9%
No Parameter Uncertainty	156,011.5	156,309.3	156,487.8	156,828.9	156,944.6	157,182.7
Actual-to-Tabular	157,364.6	158,117.2	158,571.7	159,410.9	159,718.1	160,333.2
Individual, $N = 500$	156,134.6	156,543.7	156,784.8	157,230.1	157,400.4	157,739.5
Individual, $N = 10,000$	156,048.6	156,374.6	156,569.8	156,940.2	157,076.4	157,349.6

2.4 CONCLUSION

In this case study we describe two simple ways to incorporate parameter uncertainty into mortality assumptions. We first describe how to incorporate parameter uncertainty directly into the actual-to-tabular ratio. We discuss how to optimize the MCMC estimation and the impact of incorporating parameter uncertainty on the price of a simple annuity. We also describe some of the shortcomings of our simple model and briefly discuss how to overcome them.

We then turn our attention to modeling the individual mortality rates. We show the impact of choosing different values of the prior sample size, show the impact of parameter uncertainty directly on the rates and on the price of a simple annuity and discuss the shortcomings of the model and possible model extensions.

Section 3: Health Case Study

As in other areas of insurance, predictive modeling plays a large role in health insurance. These models are used in underwriting, reserving and claims management. Many of these predictive models can be written as linear models or generalized linear models, depending on what is being predicted. A continuous response such as BMI can be modeled in a linear model as a function of variables such as weight and age. The probability an individual will lapse or renew a policy can be written as a logistic regression model using predictors such as age, sex and marital status. The relationship between the predictor variables and the outcomes may have uncertainty associated with them. Instead of using optimization methods to find point estimates and then asymptotic theory to develop distributions of the estimates, parameters uncertainty can be built into the estimation of the parameters by building a Bayesian model. We use a regression setting to compare models that incorporate parameter uncertainty with those that do not. We use a standard Bayesian regression model and a more advanced Bayesian regression model that will also incorporate uncertainty into the selection of the variables.

3.1 MODELS

3.1.1 ORDINARY LEAST SQUARES

Linear models are typically written as

$$y_j = \beta_0 + \beta_1 x_{1,j} + \beta_2 x_{2,j} + \dots + \epsilon_j,$$

where y_j is the observed data, $x_{i,j}$ are the explanatory variables, β_i represents the effect of the i -th variable on y_j and ϵ_j are independent normal error terms. The β_i coefficients represent the expected change in y_j given a unit change in $x_{i,j}$ when everything else remains unchanged.

Ordinary least squares (OLS) is a method that selects the parameters that minimize the sum of squared residuals. After the model is fit, a confidence interval can be constructed for each β_i with an upper and lower bound. Using asymptotic properties, it is even possible to construct a distribution of the parameters after the model is fit. However, no uncertainty is included in the parameters at the time the parameters are estimated, so it is estimated as if it is a static, unchanging value that we simply do not know. Also, predictions are made without any uncertainty of the parameters. The advantages of OLS are that it is conceptually simple, easy to implement and computationally fast.

3.1.2 BAYESIAN REGRESSION

In Bayesian modeling, each parameter is given a prior that reflects the modeler's belief in the distribution of possible values for the parameter. The impact of a variable may be well understood before the company data are collected or any models are fit. In that case, the prior distribution of the related parameters may be narrower (have a smaller spread). When there is limited prior knowledge, the prior distribution on the parameters will be very wide. The data will then update the prior, and the resulting distribution is called the posterior distribution. It will reflect the prior and the data, but the result is a distribution for the parameters, and it will account for uncertainty. This uncertainty will also be used in making predictions for new observations.

The regression equation in the Bayesian setting is equivalent to the OLS case. The prior distribution of the regression coefficients, β_0, β_1, \dots , will be normal (N), and the prior distribution for the variance of ϵ_i will be inverse gamma (IG). These priors are chosen for computational convenience because the math involved to update these priors using the data is straightforward (the priors are conjugate). In model form, the priors are included in the model as follows:

$$y_j = \beta_0 + \sum_{i=1}^M \beta_i x_{i,j} + \epsilon_j, \quad j = 1, \dots, n,$$

$$\beta_j \sim N(\boldsymbol{\mu}, \sigma^2 \boldsymbol{\Lambda}), \quad j = 0, \dots, M,$$

$$\epsilon_j \sim N(0, \sigma^2), \quad \sigma^2 \sim IG(a, b),$$

where $\boldsymbol{\beta}$ and $\boldsymbol{\mu}$ are vectors with $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_M)'$ and $\boldsymbol{\mu} = (\mu_0, \mu_1, \dots, \mu_M)'$, and $\boldsymbol{\Lambda}$ is a matrix with prior variance and covariance values. This matrix could be diagonal where if Λ_i represents the i th diagonal element, then the prior variance of β_i would be $\sigma^2 \Lambda_i$. The modeler will have to define the prior uncertainty by choosing values for μ_i , Λ_i , \mathbf{a} and \mathbf{b} . Each μ_i represents the prior belief in the expected value for β_i , and Λ_i is chosen to be large when there is less certainty about what the value for β_i is. The values for \mathbf{a} and \mathbf{b} should be chosen to represent the perceived variance of the response when the predictor variables are taken into account.

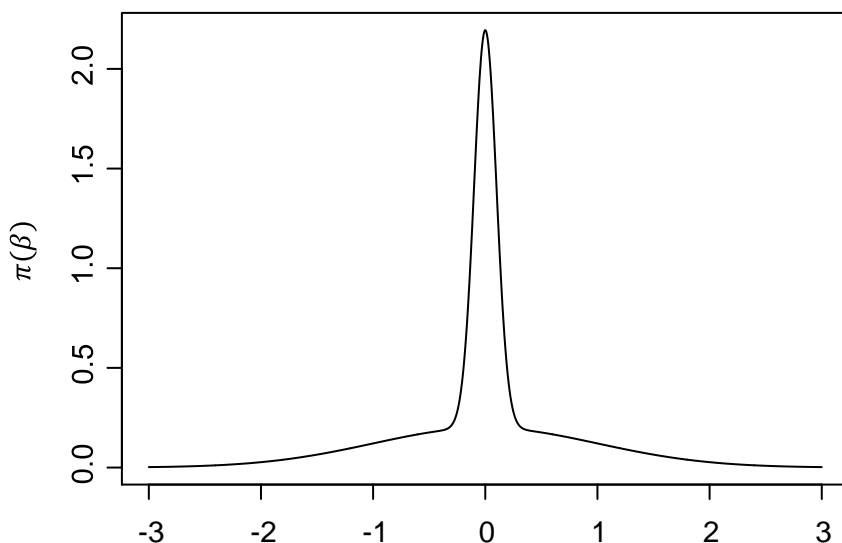
For example, suppose we are building a logistic regression model to predict whether or not an individual contracts a lung disease in the coming year. If a variable was included indicating whether or not the individual smoked, we would expect that an individual who smoked would have a higher probability of getting a lung disease, and so we would choose a prior that indicates this with a high amount of certainty. If another variable in the model was the size of the individual's house, we may not know the effect, if there is one, so we would choose a prior that is wide to reflect a high amount of uncertainty involved. We will see later how these prior values affect the estimates and prediction intervals for the parameters and predictions.

3.1.3 SPIKE AND SLAB

In the standard Bayesian regression model, the prior distribution for the coefficients is normal. When using a spike and slab prior, the prior distribution for the coefficients is a mixture of two densities, one with a small variance (spike) and one with a much wider variance (slab). Both densities are centered at zero. Such a prior will look like the density shown in Figure 11.

Figure 11

THE SPIKE AND SLAB PRIOR SHOWN IS AN EQUAL MIXTURE OF TWO NORMAL DENSITIES. BOTH HAVE MEAN 0, BUT THE VARIANCE OF ONE IS 1 AND THE VARIANCE OF THE OTHER IS 0.1.



The utility of this prior is that each β coefficient will belong to the spike component with some probability or to the slab component. Coefficients in the spike component are often deemed to be insignificant because they are so close to 0. Thus, the spike and slab model introduces a degree of uncertainty in whether or not a variable is significant. We use the following full spike and slab model:

$$y_j \sim N\left(\beta_0 + \sum_{i=1}^M \beta_i x_{i,j}, \sigma^2\right), \quad j = 1, \dots, n,$$

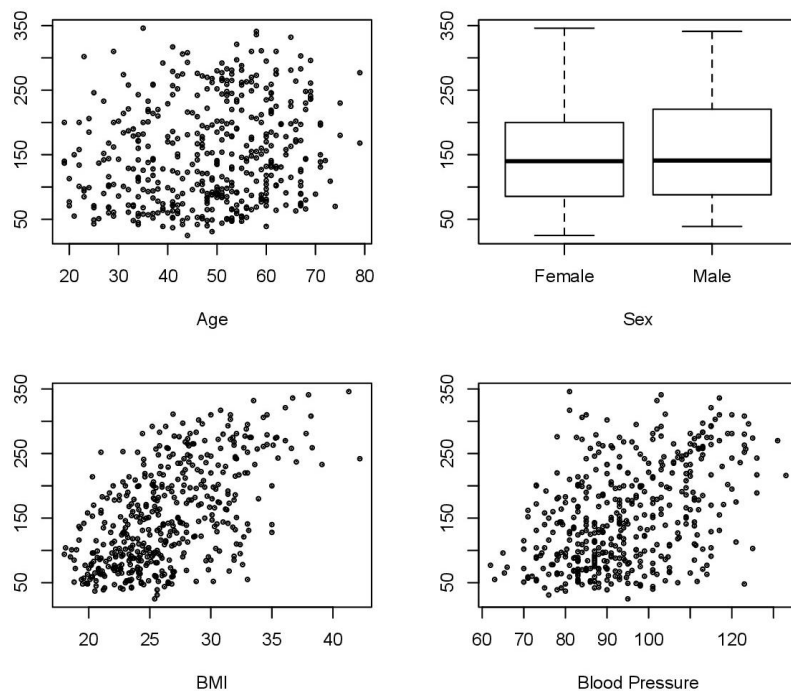
$$\beta_i \sim N(\mu_i, \tau_i^2(1\delta_{i=1} + v_0\delta_{i=0})), \quad j = 0, \dots, M,$$

$$l_i \sim \text{Bern}(w), \quad w \sim \text{Unif}(0,1),$$

$$\sigma^2 \sim IG(a_\sigma, b_\sigma), \quad \tau^2 \sim IG(a_\tau, b_\tau).$$

The power of this model comes from the assignment variable l_i which is one when the coefficient belongs to the slab and zero when the coefficient belongs to the spike, represented in the model by the Bernoulli distribution (Bern) with parameter w having a standard uniform prior distribution (Unif). The posterior distribution is drawn using Markov chain Monte Carlo (MCMC) methods, which means that a new draw for l_i is made for several iterations. If 90% of those draws are $l_i = 1$, then we could say that there is a 90% probability that the coefficient is significantly different from zero. Additional details concerning the spike and slab prior, including more specific computational methods, can be found in Ishwaran and Rao (2005).

Figure 12
SOME KEY VARIABLES ARE PLOTTED AGAINST THE RESPONSE VARIABLE FOR THE DIABETES DATA SET.



3.2 DATA ILLUSTRATION

The data we use to illustrate parameter uncertainty in health insurance contain records for 442 diabetes patients (Efron et al. (2004)). The response variable will be a measure of the progression of the disease one year after the baseline measurement was taken. The predictor variables include age sex, BMI and blood pressure as well as six different measurements of blood serum: total cholesterol (TC), low-density lipoproteins (LDL), high-density lipoproteins (HDL), the ratio between TC and HDL (TCH), lamotrigine (LTG) and glucose (GLU). Some of these key variables are plotted against the diabetes progression metric in Figure 12.

A Box-Cox power transformation is used to find the optimal transformation of the response variable. Based on this, all analysis performed uses the square root of the diabetes progression metric as the response. The relationships between these variables and the response may be more complicated than a basic linear model, so a number of quadratic terms and interactions are also included. For example, the square of age is also included in the model as a predictor, as is the product between age and each of the blood serum measurements. In all, there will be 64 predictors in the model. Each predictor is listed in

Table 3.

Table 3
LIST OF 64 PREDICTORS USED IN THE DIABETES REGRESSION ANALYSIS.

Predictors	AGE, SEX, BMI, MAP, TC, LDL, HDL, TCH, LTG, GLU
Quadratic Terms	AGE ² , BMI ² , MAP ² , TC ² , LDL ² , HDL ² , TCH ² , LTG ² , GLU ²
Interactions	AGE*SEX, AGE*BMI, AGE*MAP, AGE*TC, AGE*LDL, AGE*HDL, AGE*TCH, AGE*LTG, AGE*GLU, SEX*BMI, SEX*MAP, SEX*TC, SEX*LDL, SEX*HDL, SEX*TCH, SEX*LTG, SEX*GLU, BMI*MAP, BMI*TC, BMI*LDL, BMI*HDL, BMI*TCH, BMI*LTG, BMI*GLU, MAP*TC, MAP*LDL, MAP*HDL, MAP*TCH, MAP*LTG, MAP*GLU, TC*LDL, TC*HDL, TC*TCH, TC*LTG, TC*GLU, LDL*HDL, LDL*TCH, LDL*LTG, LDL*GLU, HDL*TCH, HDL*LTG, HDL*GLU, TCH*LTG, TCH*GLU, LTG*GLU

With so many predictors, there is sure to be many that will be insignificant in the linear model. For both the OLS model without parameter uncertainty and the Bayesian OLS model, we use an initial stepwise variable selection to determine an appropriate subset of the variables. To perform stepwise variable selection we first define some criteria to compare two models and determine which one is better. For each step of this method, we remove the one predictor that, when removed, makes the model the strongest according to the criteria we have chosen. For our model, we use Akeike's Information Criterion (AIC) as the model fit criterion to determine which variable to remove. The stepwise progression stops when all the remaining variables are significant, and removing any more will not improve the AIC.

The stepwise AIC regression results in 25 variables remaining, including sex, BMI and blood pressure. The variable age was removed, but the squared age and several interaction terms involving age still remained in the model. Each of the serum measurements is used at least once in one of the interactions. The TCH and GLU serum measurements are the least present in the remaining predictors.

The prior uncertainty we choose can reflect scientific knowledge that relates the predictors with diabetes progression. For example, a higher BMI should be positively correlated with diabetes progression, so the prior for the regression coefficient of BMI should be positive. To help in understanding the interpretation of the priors, the variables are all standardized by subtracting the mean of the variable and dividing by the standard deviation. This means that if β_j is the coefficient for variable X_j , then a one standard deviation change in X_j results in a β_j standard deviation change in Y , when everything else is held constant. All the priors for the coefficients are chosen based on the perceived positive or negative effect on diabetes. A list of the priors chosen for each of the coefficients is shown in the following table.

Table 4
THE PRIOR MEANS AND VARIANCES FOR EACH OF THE UNKNOWN COEFFICIENTS IN THE REGRESSION MODEL. THE DISTRIBUTION OF EACH OF THE PRIORS IS NORMAL.

Variable	Prior Mean	Prior Variance
Age	0	4
Sex	-0.5	1
BMI	1	1
BP	1	1
TC	1	1
LDL	-1	1
HDL	-1	1
TCH	0	4
LTG	0.5	1
GLU	1	1
Everything Else	0	9

The priors for the spike and slab model are defined hierarchically as in Section 3.1.3, with $a_\sigma = b_\sigma = a_\tau = b_\tau = 0.01$. These are noninformative priors, which means that we use very little prior information for the variance terms. The spike term is set to $v_0 = 0.005$, which was determined by exploring several options and comparing the resulting fit.

To compare the goodness of fits between the three models, a test sample was left out of the data set. The parameters were estimated with 400 observations, and 42 were left out to predict (stepwise AIC variable selection was performed on the 400 remaining in the data set). The mean square prediction error (MSPE) is the squared sum of prediction errors divided by the number of predictions. The MSPE for the OLS model was 4.28. For the Bayesian OLS model, the MSPE was 3.88. The MSPE for the spike and slab model was 3.50. The lower values indicate that accounting for uncertainty improves the predictive power of the model.

The Bayesian model with the spike and slab prior estimates inclusion probabilities from the entire set of variables. Table 5 compares the inclusion probability from the spike and slab model to whether or not the stepwise AIC regression model includes the variables.

Table 5
INDICATORS FOR THE INCLUSION IN THE STEPWISE AIC VARIABLE, WHERE 1 INDICATES THAT IT WAS INCLUDED IN THE FINAL MODEL AND 0 INDICATES THAT IT WAS NOT, COMPARED AGAINST THE INCLUSION PROBABILITY WHEN USING A SPIKE AND SLAB PRIOR.

Variable	Stepwise Inclusion	Spike and Slab Inclusion
Age	0	0.517
Sex	1	0.985
BMI	1	1
BP	1	0.998
TC	1	0.451
LDL	1	0.511
HDL	1	0.97
TCH	0	0.724
LTG	0	1
GLU	0	0.645

The two methods of determining the significance of inclusion have many similarities but some key differences. For example, sex, BMI and blood pressure are highly significant in the spike and slab model and are also included in the stepwise regression. However, LTG is highly significant in the spike and slab model and yet was not included when using the stepwise regression.

We can compare the confidence intervals of the predictions between the three models to understand how parameter uncertainty affects the width. The average interval width was 8.84 for the standard regression model, 9.07 for the Bayesian regression model and 8.90 for the spike and slab model. Adding uncertainty in the parameters increases the interval width of the prediction slightly, as would be expected with extra uncertainty. The spike and slab model actually has a smaller interval width than the Bayesian regression model. One possible explanation is that in the spike and slab model, variables that are mildly significant are included in the model only sometimes, decreasing the variability from these variables, whereas in the standard Bayesian OLS, all variables are included, even if the significance is slight.

An important concept in Bayesian modeling is prior sensitivity. Running the same model with different priors can result in different posterior distributions. A poorly chosen prior with too much certainty can cause an inappropriate amount of bias in the posterior. To test this in the diabetes data, we choose three sets of priors for σ^2 and τ^2 and examine the effect in the posterior. In all three cases the prior is the same for the two parameters. The priors are IG(.01,.01), IG(5,5) and IG(100,100). Table 6 summarizes the posterior means of the two parameters under the different prior settings.

Table 6
MEANS OF THE POSTERIOR DISTRIBUTION FOR THE PARAMETERS σ^2 AND τ^2 UNDER DIFFERENT PARAMETERS SETTINGS.

Prior	Posterior Mean of σ^2	Posterior Mean of τ^2
IG(.01, .01)	0.25	0.05
IG(5, 5)	0.37	0.15
IG(100, 100)	0.93	0.75

The IG(.01,.01) prior is a very noninformative prior, which suggests a lot of uncertainty. The IG(100,100) is very informative, which suggests that the modeler is fairly certain about what the distribution of the parameters should be. Clearly, from the analysis, this was much too certain because the posterior means, and therefore the posterior distribution, is very far from the other priors. This situation would be one where the modeler was certain about a particular parameter but was very wrong. When information is known to some extent, there should be a balance where the prior is well thought out with respect to whatever information the modeler has while not imbuing more certainty than is necessary.

3.3 PREDICTION COMPARISON

The amount of uncertainty included in the parameters can influence both the parameter estimates and predictions for new observations. Even more so, the intervals of these estimates and predictions can be influenced by the amount of parameter uncertainty. This can be controlled in a Bayesian regression model using different choices for prior values. We will first address the effect of the priors on the parameter estimates and credible interval and then the effect of the prior on the prediction of a new observation followed with the associated credible interval.

3.3.1 PARAMETER ESTIMATES

To explore the effect of the prior on parameter estimates, we will single out the BMI variable in the diabetes data set. We will work with the Bayesian regression model given in section 3.1 and specifically change the values of μ_i and Λ_i that are associated with the BMI parameter. We will keep all other prior values the same and change only these two and see how they affect the parameter estimate and associated credible interval. BMI should be positively associated with the diabetes index, so a reasonable prior for the coefficient associated with BMI could be $\mu_i = 0.5$, which would suggest that in the prior, a one standard deviation change in BMI results in an average change in the response of 0.5 standard deviations. For illustration, we will also use a poor choice for a prior of $\mu_i = -0.5$ which suggests that higher BMI actually reduces the degree that diabetes progresses. We will also use three different values of the spread, allowing Λ_i to be a reasonable value of 10, a small value of 0.01, or a large value of 10,000. The OLS estimate for the coefficient, β_i , that corresponds to BMI was 23.06. Table 7 reports the estimates and confidence intervals under parameter uncertainty with different priors. Although the analysis was performed with standardized variables as described in section 3.2, the values in Table 7 are scaled back to original scale.

Table 7
ESTIMATES AND 95% CREDIBLE INTERVALS FOR THE BMI COEFFICIENT USING DIFFERENT PRIOR SETTINGS

Prior	Estimate of β_i	Credible Interval	Credible Interval Width
$\mu_i = 0.5, \Lambda_i = 10$	23.05	(19.30, 27.11)	7.81
$\mu_i = 0.5, \Lambda_i = .01$	24.88	(21.68, 28.33)	6.65
$\mu_i = 0.5, \Lambda_i = 10,000$	23.06	(19.18, 27.30)	8.12
$\mu_i = -0.5, \Lambda_i = 10$	23.04	(19.12, 27.08)	7.96
$\mu_i = -0.5, \Lambda_i = .01$	7.54	(3.70, 11.50)	7.81
$\mu_i = -0.5, \Lambda_i = 10,000$	23.06	(19.08, 27.18)	8.10

As we can see from the table, the estimate changes significantly when the prior variance is smaller but doesn't change much when it is larger. Also, because the size of the data set is large enough, choosing a poor prior mean makes the largest impact with a small variance. However, choosing the prior variance to be larger than really was necessary resulted in a larger credible interval. If tighter intervals are important, it is best to strike a balance in the prior variance and not choose one that is too small so that the prior mean carries too much weight or too large, unnecessarily putting extra uncertainty into the model.

3.3.2 PREDICTIONS OF NEW OBSERVATIONS

We can repeat this study for predictions of new observations. The prior values of a and b play an important role in determining the estimates and intervals of the predictions of new observations. To be clear, the prior expected value of the variance of observations is $\frac{b}{a-1}$, so as b increases relative to a , the prior variance of the observations increases, and the prediction interval would increase. As a and b both increase, the weight given to the prior will increase as well. For exploration of this we will again use the diabetes data set with a new observation with all variables equal to the 41st record in the data set. This new individual is a 50-year-old male with a BMI of 25.6 and a BP of 101. We will explore the value of the estimate and the interval of the new observation using different priors for a and b . Specifically, we will allow a to be a reasonable value of 2.5 and then a large value of 100, and then we will allow b to also be both 2.5 and 100. For reference, the OLS estimate was 12.49 with an interval of (8.19,16.79), which has a length of 8.59.

Table 8
ESTIMATES AND 95% CREDIBLE INTERVALS FOR THE PREDICTION OF A NEW OBSERVATION USING DIFFERENT PRIOR SETTINGS

Prior	Estimate of β_i	Credible Interval	Credible Interval Width
$a = 2.5, b = 2.5$	12.30	(7.91, 16.47)	8.56
$a = 2.5, b = 100$	12.13	(6.25, 18.18)	11.94
$a = 100, b = 2.5$	12.22	(8.59, 15.91)	7.32
$a = 100, b = 100$	12.20	(6.85, 17.25)	10.41

Although the estimate was not affected much by the change in prior values, the intervals and interval widths did change. We can thus affect how much uncertainty there is in our predictions by how we choose our priors. If we are certain that the prior variance should be high, it would be best to choose large values for a and b . If we are certain that the variance is low, it would be best to choose a small value for b and a large value for a . If there is not much information to determine, we should choose small values for both, but we can still fine-tune it according to what beliefs we may have, such as making b larger than a or vice versa.

While this exploration was carried out for the standard Bayesian regression, the principles hold true for the corresponding parameters in the spike and slab model.

3.4 CONCLUSION

The diabetes data set illustrates how parameter uncertainty can affect prediction in a health-related example. Even though this is just one specific application, the principles of modeling with parameter uncertainty that was used here can be used in a number of other settings where a health outcome is being predicted by a set of variables. Adding the uncertainty helped to predict more accurately in this case. The prediction intervals were wider in general when extra uncertainty was added. These are both observations that can be seen regularly when adding uncertainty in the parameters.

In the Bayesian regression model, uncertainty in the parameters was specified by logic and reasoning, using facts that are already known in the health community. Details to inform the prior such as these may be present in other health situations as well. When information is not available, wide priors can be used to express a great amount of uncertainty. In the spike and slab model, there is uncertainty in the parameter values but also uncertainty in the significance of the parameters. This adds extra strength to the model because it allows the predictions to be made with a different set of predictors chosen using probabilistic methods. This extra strength was seen when the spike and slab model had the lowest MSPE in the diabetes data set.

Section 4: Property/Casualty Case Study

This study seeks to explore the differences and similarities between a frequentist and a Bayesian approach to parameter uncertainty when applied to a property and casualty situation. This paper will use claim counts in California in 1999 as an example situation and as a basis for this study.

4.1 DATA

Our data contain the claim history of 79,079 policyholders. More than 60% of the policyholders had zero claims, but high claim counts by other policyholders (potentially commercial or fleet clients) drove the mean claim count up to 10.26. In addition, the data set included the following descriptive information for each insured:

- Program (PRG). There were five options for possible programs available to an insured: Standard (denoted with program code 12), Substandard (program code 3), Assigned Risk (program code 4), Direct Excess (program code 5) and Motorcycle (program code 6). As to be expected, the two largest categories are Standard (35.35% of the data set) and Substandard (31.53%).
- Coverage (COV). This category was broken into three subcategories: Bodily Injury (coverage code 1), Property Damage (coverage code 2) and Combined Single Limits (coverage code 6). With only 7.36% of insurance policies using Combined Single Limits coverage, the rest of the insureds split the remaining two categories almost evenly.
- Limit (LMT). Like the previous category, Limit was also divided into three subsections: Basic (limit code 1), Greater Than Basic (limit code 2) and None Given (limit code 0). Nearly half (45.13%) of the insureds listed had a Greater Than Basic limit on their policy.
- Driver Status (DVR). The three subcategories for this section are as follows: Good Driver Discount Applied (driver code 1), Good Driver Discount Not Applied (driver code 2) or None Given (driver code 0). At 35.24%, the largest group of insureds in the data set had a driver code of 0, potentially the commercial clients mentioned above.
- Zip Code (ZIP). This section listed where an insured lived by zip code. Our data set includes only the state of California.
- Income per Household (IPH): At the zip code level, this is the average income per household.

In addition to the above information, the data set also included the written and earned premiums and exposures. We removed any observations with negative claims or exposures. We also dropped those with a median household income of 0.

4.2 MODELS

A Poisson regression model is typically written as

$$y_j \sim \text{Poisson}(\lambda_j),$$

$$\log(\lambda_j) = \beta_0 + \beta_1 x_{1,j} + \beta_2 x_{2,j} + \dots + \beta_p x_{p,j},$$

where $\log(\lambda_j)$ is called the link function, $x_{1,j}, \dots, x_{p,j}$ are the p predictor variables for each record j and y_j is the j th observed claim count. We first fit the data using a frequentist Poisson regression, including an intercept term, PRG, DVR, COV, LMT and IPH. Then we compare those results to a Bayesian Poisson regression with normal priors on the coefficients, specifically

$$\beta_i \sim N(\mu_i, \sigma_i).$$

Each μ_i represents the prior belief of the effect of the i th predictor on the expected log claim counts. The σ_i is a measure of confidence in that prior estimate. When σ_i is large, the prior will not affect the results because there will be a lot of prior uncertainty. When σ_i is small, the prior may significantly alter the results because there is little uncertainty in the prior as to the effect of the corresponding predictor. The effect of this prior uncertainty on the estimates and predictions will be explored in the analysis of the data. Results for negative binomial regression in the Bayesian setting are also included to address the issue of overdispersion. We use the log link in the negative binomial regression model as well. As is common in models for claim counts, earned exposure was included as an offset.

4.3 RESULTS

4.3.1 POISSON REGRESSION

In an attempt to explore the impact of having a prior distribution on each of the parameters, we adjusted the standard deviation of each prior to reflect a different level of confidence in our prior distribution. Large standard deviations suggest more uncertainty in the prior distribution and allow for the data to have a stronger impact on the model, while a small standard deviation diminishes the impact of the data on the prior belief, which means that more data would be required to affect the posterior. In all cases the mean of the prior distribution is 0, so a small standard deviation would mean that we believe the coefficient is not likely to be significantly different from zero, or, equivalently, that we require strong evidence before we say that the coefficient is significant.

Using R and Stan, we fit the model to the data and obtain the results in Table 9. When we decrease the standard deviation, the parameter estimates are pulled toward zero, reducing their impact on the model. We did not alter the prior distribution for β_0 (the intercept term) from normal(0,100) because it does not make sense to shrink the intercept to zero because you will be dropping the intercept from the model. The last column of the table is the estimates from the frequentist Poisson regression.

Table 9
POSTERIOR PARAMETER ESTIMATES FOR DIFFERENT VALUES OF σ USING POISSON REGRESSION

	σ							
	0.0001	0.001	0.1	1	30	50	500	Freq.
Intercept	-6.04	-6.11	-6.16	-6.15	-6.15	-6.15	-6.15	-6.15
PRG=4	0.00	0.00	0.40	0.40	0.40	0.40	0.40	0.40
PRG=5	0.00	0.00	-2.37	-4.52	-4.67	-4.67	-4.68	-4.66
PRG=6	0.00	-0.02	-2.27	-2.38	-2.38	-2.38	-2.38	-2.38
PRG=12	0.00	-0.03	-0.20	-0.21	-0.21	-0.21	-0.21	-0.21
DVR=1	0.00	-0.05	0.01	0.02	0.02	0.02	0.02	0.02
DVR=2	0.00	0.50	0.44	0.44	0.44	0.44	0.44	0.44
COV=2	0.00	0.20	1.10	1.10	1.10	1.10	1.10	1.10
COV=6	0.00	0.00	-0.18	-0.19	-0.19	-0.19	-0.19	-0.19
LMT=1	0.00	-0.02	-0.45	-0.46	-0.46	-0.46	-0.46	-0.46
LMT=2	0.00	0.02	-0.49	-0.49	-0.49	-0.49	-0.49	-0.49
IPH	0.00	0.00	0.03	0.03	0.03	0.03	0.03	0.03

In addition to fitting the model, the Bayesian method also creates posterior distributions on each of the parameters once they have been updated by the data. Table 10 provides the length of the 95% credible intervals for each coefficient by the size of σ . Similar to the parameter estimates, the length of the credible intervals was really small when σ was small but rapidly stabilized as σ grew.

Table 10
LENGTH OF PARAMETER CREDIBLE INTERVALS FOR DIFFERENT VALUES OF σ USING POISSON REGRESSION

	σ								
	0.0001	0.001	0.1	1	30	50	100	500	1,000
Intercept	0.32	0.54	2.35	2.44	2.46	2.49	2.45	2.5	2.47
PRG=4	0.03	0.34	4.22	3.49	3.86	3.73	4.08	3.69	3.76
PRG=5	0.03	0.32	15.6	51.93	64.52	65.27	63.67	63.4	60.57
PRG=6	0.03	0.32	6.33	7.3	7.46	7.27	7.08	7.8	7.18
PRG=12	0.03	0.32	1.1	1.03	1.04	0.99	1.14	1.1	1.12
DVR=1	0.03	0.29	1.04	1	1.08	1.05	1	0.98	0.98
DVR=2	0.03	0.33	1.25	1.17	1.21	1.26	1.16	1.17	1.22
COV=2	0.03	0.29	0.79	0.76	0.7	0.76	0.81	0.76	0.78
COV=6	0.03	0.31	5.68	5.89	6.08	5.84	5.75	5.94	6.1
LMT=1	0.03	0.29	2.69	2.53	2.71	2.67	2.54	2.53	2.57
LMT=2	0.03	0.31	2.44	2.49	2.69	2.57	2.47	2.45	2.63
IPH	0.03	0.23	0.33	0.37	0.34	0.34	0.35	0.34	0.34

To further illuminate the effect of σ , we estimate the prediction intervals for a standard observation in our data set (using the most likely value of each categorical observation and the mean of IPH). Table 11 includes the 95% prediction intervals as well as the spread between the upper and lower bounds of the interval.

Table 11
 POSTERIOR PREDICTIVE RANGES FOR DIFFERENT VALUES OF σ USING POISSON REGRESSION. PREDICTION FOR AN OBSERVATION WITH THE MOST COMMON VALUES OF THE CATEGORICAL VARIABLES AND THE MEAN VALUE OF IPH

σ	Lower	Upper	95% Spread
0.0001	0.829	0.839	0.01
0.001	0.838	0.849	0.01
0.1	0.418	0.429	0.011
1	0.342	0.36	0.018
30	0.338	0.356	0.019
50	0.338	0.356	0.018
100	0.338	0.356	0.019
500	0.338	0.356	0.019
1,000	0.337	0.356	0.019

The first two confidence intervals are estimates essentially around the intercept because all the regression coefficients are essentially zero. The estimates are unduly influenced by the prior leading to an inaccurate model fit. Once σ grows to a reasonable size, the coefficient estimates stabilize (especially those near zero), making the prediction intervals very similar as well.

It is important to see how parameter uncertainty can affect prediction of claim amounts. We will illustrate this by using the claim count data set and seeing how well in-sample prediction is performed for the mean response. For example, if a certain individual in the data set has made 30 claims, we will use that individual’s covariate values in the model with estimated parameters and predict the mean number of claims that individual would make. We will see how accurate the estimate is by assessing the average bias, or difference from the estimate to the truth, and also how wide the confidence interval is. To see how prediction compares when claims counts are high versus low, we will predict two groups: all individuals in the data set who have made one claim and all individuals who have made 30 claims. Table 12 summarizes the relevant values in this scenario. The model with parameter uncertainty used a prior variance of 10 for the parameters.

Table 12
 COMPARING PREDICTION BIAS AND CONFIDENCE INTERVAL WIDTH BETWEEN MODELS WITH AND WITHOUT PARAMETER UNCERTAINTY

	Freq		Uncertainty	
	1 Claim	30 Claims	1 Claim	30 Claims
Bias	1.44	26.65	0.948	26.14
CI Width	0.038	0.015	0.067	0.088

Note that the confidence intervals are small because these predictions are for the mean of the Poisson distribution for claim counts. These are not predictions for an individual. To get predictions for an individual, we would take the means in these predictions and find the confidence interval using a Poisson distribution with these predicted means. From this table we can see that including parameter uncertainty does have a slight benefit in prediction accuracy, especially at the lower claim counts. Predicting the higher claim counts proves to be difficult for either setting, but since the variance of the Poisson distribution is quite large at high values, seeing the higher bias for large claims is not an indication of a poor model. What is interesting, however, is that at higher claim counts, where the prediction accuracy is poorer, the confidence interval width under the frequentist model is smaller than at lower claim counts. The model with parameter uncertainty has a wider confidence interval at higher claim counts. This may be due to certain parameters indicating higher claim counts and how much uncertainty was included in those parameters. This procedure was repeated for several different prior values for the variance, and although the predictions varied slightly, the conclusions remained the same.

4.3.2 NEGATIVE BINOMIAL REGRESSION

The analysis from section 4.3.1 is repeated using a negative binomial likelihood. This is a common approach when it is believed that the data may be overdispersed, meaning that the variance of predictions should not be restricted to be equal to the mean but should be greater than the mean instead. Because a log-link function is used in negative binomial regression, the interpretation of parameter values is similar to the Poisson regression model, so parameter estimates will typically be similar. Table 13 shows parameter estimates under the negative binomial regression model. This can be directly compared to Table 9, where most of the estimates are similar.

Table 13
POSTERIOR PARAMETER ESTIMATES FOR DIFFERENT VALUES OF σ USING NEGATIVE BINOMIAL REGRESSION

	σ							
	0.0001	0.001	0.1	1	30	50	500	1,000
Intercept	-6.03	-6.03	-5.94	-5.91	-5.91	-5.91	-5.91	-5.91
PRG=4	0.00	0.00	0.46	0.47	0.47	0.47	0.47	0.47
PRG=5	0.00	0.00	-2.32	-4.57	-4.73	-4.74	-4.74	-4.76
PRG=6	0.00	-0.01	-2.34	-2.49	-2.49	-2.49	-2.49	-2.49
PRG=12	0.00	0.00	-0.39	-0.42	-0.42	-0.42	-0.42	-0.42
DVR=1	0.00	0.00	-0.10	-0.09	-0.09	-0.09	-0.09	-0.09
DVR=2	0.00	0.00	0.26	0.29	0.29	0.29	0.29	0.29
COV=2	0.00	0.01	0.96	0.96	0.96	0.97	0.96	0.96
COV=6	0.00	0.00	-0.09	-0.10	-0.10	-0.10	-0.10	-0.10
LMT=1	0.00	0.00	-0.39	-0.43	-0.43	-0.43	-0.43	-0.43
LMT=2	0.00	0.00	-0.40	-0.42	-0.42	-0.42	-0.42	-0.42
IPH	0.00	0.00	0.04	0.04	0.04	0.04	0.04	0.04

Although estimates will be similar, because the negative binomial includes more uncertainty for the responses than the Poisson likelihood, the variance in the parameter estimates will be significantly reduced. When compared to the Poisson model, there is a shift in where the uncertainty is, but overall uncertainty of predictions will remain relatively constant. Tables 14 and 15 can be compared to Tables 10 and 11, but using the negative binomial regression instead of Poisson regression. As is typical, the credible intervals for the parameter estimates are significantly smaller, but the predictions have similar spreads.

Table 14
LENGTH OF PARAMETER CREDIBLE INTERVALS FOR DIFFERENT VALUES OF σ USING NEGATIVE BINOMIAL REGRESSION

	σ								
	0.0001	0.001	0.1	1	30	50	100	500	1,000
Intercept	0.0143	0.0143	0.0500	0.0493	0.0474	0.0503	0.0490	0.0479	0.0499
PRG=4	0.0003	0.0030	0.0816	0.0832	0.0890	0.0807	0.0856	0.0843	0.0872
PRG=5	0.0003	0.0033	0.0690	0.5660	0.6178	0.6147	0.6236	0.5968	0.6273
PRG=6	0.0003	0.0035	0.0804	0.0820	0.0803	0.0831	0.0778	0.0803	0.0830
PRG=12	0.0003	0.0033	0.0302	0.0303	0.0306	0.0313	0.0309	0.0319	0.0309
DVR=1	0.0003	0.0032	0.0398	0.0378	0.0377	0.0407	0.0376	0.0369	0.0399
DVR=2	0.0003	0.0033	0.0416	0.0377	0.0403	0.0416	0.0383	0.0411	0.0405
COV=2	0.0003	0.0031	0.0303	0.0301	0.0291	0.0311	0.0300	0.0301	0.0292
COV=6	0.0003	0.0003	0.0825	0.0823	0.3982	0.0779	0.0805	0.0810	0.0893
LMT=1	0.0003	0.0003	0.0566	0.0534	0.0531	0.0563	0.0541	0.0536	0.0589
LMT=2	0.0001	0.0030	0.0533	0.0520	0.0518	0.0558	0.0538	0.0517	0.0590
IPH	0.0003	0.0032	0.0142	0.0150	0.0153	0.0156	0.0150	0.0148	0.0147

Table 15

POSTERIOR PREDICTIVE RANGES FOR DIFFERENT VALUES OF σ USING NEGATIVE BINOMIAL REGRESSION. PREDICTION FOR AN OBSERVATION WITH THE MOST COMMON VALUES OF THE CATEGORICAL VARIABLES AND THE MEAN VALUE OF IPH

σ	Lower	Upper	95% Spread
0.0001	0.840	0.854	0.015
0.001	0.837	0.851	0.015
0.1	0.440	0.453	0.013
1	0.359	0.377	0.019
30	0.353	0.373	0.020
50	0.354	0.372	0.019
100	0.353	0.373	0.020
500	0.353	0.373	0.021
1,000	0.353	0.372	0.019

4.4 CONCLUSION

In this section, we have shown a simple way to incorporate parameter uncertainty in a Poisson regression. There are many ways to incorporate parameter uncertainty in property and casualty insurance. For some examples, see Frees, Shi and Valdez (2009) or Sriram, Shi and Ghosh (2016), among many others. A simple search for Bayesian claim count modeling will yield many results.

Section 5: Conclusion

In this report we examine the impact of parameter uncertainty in many different contexts. In Section 1, we review the current literature on parameter uncertainty, especially with an eye toward their widely varying applications. We then move into three case studies presenting simple ways to incorporate parameter uncertainty in life, health and property and casualty insurance. In life insurance, we examine how to incorporate parameter uncertainty in both the actual-to-tabular ratio and the individual mortality rates. We discuss the limitations of the models and potential extensions to solve those limitations. In health insurance, we show how to incorporate parameter uncertainty when choosing the explanatory variables to include in the model. We apply that methodology to a diabetes data set. In property and casualty insurance, we compare Poisson and negative binomial regression models with and without parameter uncertainty and apply the models to a data set of claim counts in California.

Section 6: References

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